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ON VECTOR DESCRIPTION OF ARBITRARY DEFORMATION OF SHELLS

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Abstract—The purpose of the present note is to remove the restriction of linearity of deformation analysis from the vectorial kinetostatics relations of shell theory[1, 2]. The resulting nonlinear vector compatibility equations have a prototype in the moment stress elasticity[3].

1. INTRODUCTION

Local form of a shell reference surface is described with four vectors: two curvature vectors (Darboux) and two derivatives of the position vector with respect to surface coordinates. This approach assures a short-cut to the analysis of the local geometry of a surface as well as to the derivation formulas. It was in fact suggested in Love's book.

A change of the four vector parameters, caused by deformation, implies four parameters of strain, connected by two nonlinear vector compatibility equations.

2. GEOMETRY OF A SURFACE

Consider a surface described by the equation $\mathbf{r} = \mathbf{r}(\xi_1, \xi_2)$. The derivatives $\mathbf{r}_{,j}$ with respect to the coordinates ξ_j determine the unit tangent vectors $\mathbf{t}_j$ (Fig. 1) of the ξ_j-lines, i.e. $\mathbf{r}_j = \alpha_j \mathbf{t}_j$. The angle $\chi = \chi(\xi_j)$ between the ξ_j-lines may be oblique and variable. During a deformation of the surface, which may be arbitrary, each particle retains its curvilinear coordinates ξ_j. The characteristics of the deformed surface will be denoted by asterisk ($\mathbf{r}^*, \mathbf{t}_j^*, \chi^*$, etc.). The undeformed state is to be considered as a particular case of the deformed one.

The curvature of the surface will be measured by the rotation of a tangent plane sliding along the surface. The position of the plane is identified by a (unit) vector normal to the surface $\mathbf{n}^* = \mathbf{t}_1^* \times \mathbf{t}_2^* / \sin \chi^*$. To measure its rotation around $\mathbf{n}^*$ the plane must be connected with an element of the surface. (Because of the shear deformation it cannot be more than one linear element at a point of the surface.) We choose for this an element $\mathbf{t}^* ds$, $\mathbf{t}^* = (\mathbf{t}_1^* + \mathbf{t}_2^*)/(|\mathbf{t}_1^* + \mathbf{t}_2^*|$ bisecting the angle χ^* between the coordinate lines (Fig. 1).

We introduce two vector parameters $\mathbf{k}_1^*$, $\mathbf{k}_2^*$ of surface curvature, defining $\mathbf{k}_1^* d\xi_1 + \mathbf{k}_2^* d\xi_2$ as an angle ($d\mathbf{\Phi}$) between tangent planes at points $M(\xi_1, \xi_2)$ and $M_1(\xi_1 + d\xi_1, \xi_2 + d\xi_2)$.

The geometric meaning of the curvature vectors $\mathbf{k}_1^*$, $\mathbf{k}_2^*$ becomes more lucid with their component representation in the following two presentations identical to each other before deformation:

$$\frac{\mathbf{k}_i^*}{\alpha_i^*} = \frac{\mathbf{n}^* \times \mathbf{t}_j^*}{R_{ij}^*} + \frac{\mathbf{n}^*}{R_{i3}^*} \quad \text{or} \quad \frac{\mathbf{k}_i^*}{\alpha_i} = \frac{\mathbf{n}^* \times \mathbf{t}_j'}{R_{ij}'} + \frac{\mathbf{n}^*}{R_{i3}'} \quad (i, j = 1, 2). \tag{1}$$

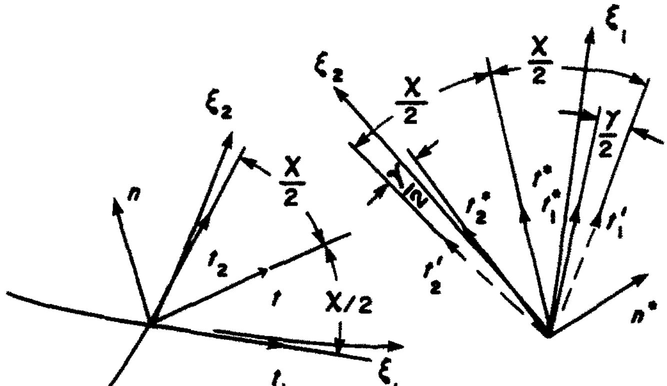

Fig. 1.

The unit vectors t'_j are defined as indicating the position attained by the vectors $\mathbf{t}_j$, which when rotating during the course of the shell deformation are rigidly connected to the tangent plane, i.e. to the vectors $\mathbf{n}^*$ and $\mathbf{t}^*$. The difference between $\mathbf{t}'_j$ and $\mathbf{t}^*_j$ is determined solely by the shear angle $\gamma = \chi - \chi^*$. The vectors $\mathbf{t}'_j$ remain orthogonal, when the coordinate lines ξ_i are orthogonal before the deformation. (The use of such an orthogonal auxiliary basis was suggested by Simmonds and Danielson[4]).

The expressions of $\mathbf{t}'_j$ through $\mathbf{t}^*_j$ are readily written out; and are for small strain ($/\gamma/ \ll 1$) and $\chi = (\pi/2)$:

$$\mathbf{t}'_1 = \mathbf{t}^*_1 - \frac{\gamma}{2}\mathbf{t}^*_2, \quad \mathbf{t}'_2 = \mathbf{t}^*_2 - \frac{\gamma}{2}\mathbf{t}^*_1. \tag{2}$$

The components of the curvature vectors $\mathbf{k}^*_j$ as introduced by eqns (1) are easily recognized to be the normal section curvatures ($1/R^*_{ii}$), the twist ($1/R^*_{12}$, $1/R^*_{21}$) and the in-plane curvatures of the ξ-lines ($1/R^*_{i3}$). The curvature parameters $\mathbf{k}^*_j$ render derivation formulas for all the basis vectors introduced. Considering an auxiliary unit vector $\mathbf{v}^*(\xi_i)$ directed at any point $M(\xi_i)$ at the same angles to $\mathbf{n}^*(\xi_i)$ and $\mathbf{t}^*(\xi_i)$ the definition of $\mathbf{k}^*_j$ means that $\mathbf{k}^*_j\, d\xi_j$ is the angle between $\mathbf{v}^*(\xi_i)$ and the vector $\mathbf{v}(\xi_j + d\xi_j)$ in the adjacent point $M(\xi_j + d\xi_j)$.
This amounts to

$$\mathbf{v}^*(\xi_j + d\xi_j) = \mathbf{v}^*(\xi_j) + \mathbf{k}^*_j d\xi_j \times \mathbf{v}^*(\xi_j).$$

Hence the expressions for the derivatives of a unit vector $\mathbf{v}^*$ or its particular cases $\mathbf{n}^*$, $\mathbf{t}^*$ are:

$$\mathbf{v}^*_{,j} = \mathbf{k}^*_j \times \mathbf{v}^*, \quad \mathbf{n}^*_{,j} = \mathbf{k}^*_j \times \mathbf{n}^* \quad \text{and} \quad \mathbf{t}^*_{,j} = \mathbf{k}^*_j \times \mathbf{t}^*. \tag{3}$$

The position of the vectors $\mathbf{t}'_i$ or $\mathbf{t}^*_i$ changes inside the tangent plane together with the angles $(-1)^i\mathbf{n}^*\chi(\xi_j)/2$ or $(-1)^i\mathbf{n}^*\chi^*(\xi_j)/2$ between $\mathbf{t}^*$ and $\mathbf{t}'_i$ or t^*_i (Fig. 1). Taking account of this the derivation formulas for t'_i and t^*_i contain, compared to (3), additional terms:

$$\mathbf{t}'_{i,j} = \left[\mathbf{k}^*_j + \frac{1}{2}(-1)^i\chi_{,j}\mathbf{n}^*\right] \times \mathbf{t}'_i, \quad \mathbf{t}^*_{i,j} = \left[\mathbf{k}^*_j + \frac{1}{2}(-1)^i\chi^*_{,j}\mathbf{n}^*\right] \times \mathbf{t}^*_i. \tag{4}$$

The formulas (3) lead to a compatibility equation between the two vector parameters $\mathbf{k}^*_j$. Applying (3) to the equation $\mathbf{v}^*_{,21} = \mathbf{v}^*_{,12}$ and using the vector triple product expansion formula results in

$$\mathbf{k}^*_{1,2} - \mathbf{k}^*_{2,1} + \mathbf{k}^*_1 \times \mathbf{k}^*_2 = 0. \tag{5}$$

This is equivalent to the three scalar Gauss–Codazzi equations.

With the derivation formulas (4) the equation $\mathbf{r}_{,12} = \mathbf{r}_{,21}$ renders the relation $R_{12} = R_{21}$ and the well-known expressions of geodetic curvatures $1/R_{i3}$ with Lame's parameters α_j.

The description of local geometry is completed by formulas for the curvature parameters $\mathbf{k}_j^\alpha$ for any coordinate system ξ_i^α defined by its angle α with the original system ξ_i. This includes formulas for R^α_{ij}, their extreme values, invariants, etc.

These formulas follow directly from $\mathbf{k}_i^\alpha d\xi_i^\alpha = \mathbf{k}_1 d\xi_1 + \mathbf{k}_2 d\xi_2$, which is just another way to write the relation $d\mathbf{\Phi} = \mathbf{\Phi}_{,1} d\xi_1 + \mathbf{\Phi}_{,2} d\xi_2$.

3. STRAIN AND COMPATIBILITY

It is natural to describe the deformation of the surface with parameters ($\boldsymbol{\kappa}_i$, $\boldsymbol{\epsilon}_i$) reflecting the change of the local geometry characteristics from $\mathbf{k}_i$, $\mathbf{r}_{,i}$ to $\mathbf{k}^*_i$, $\mathbf{r}^*_{,i}$ in the simplest way:

$$\mathbf{k}^*_i = \mathbf{k}_{iR} + \alpha_i\boldsymbol{\kappa}_i, \qquad \mathbf{r}^*_{,i} = (\mathbf{r}_{,i})_R + \alpha_i\boldsymbol{\epsilon}_i. \tag{6}$$

Additionally we require the strain parameters $\boldsymbol{\kappa}_i$, $\boldsymbol{\epsilon}_i$ to be equal to zero at any point, at which the surface is not deformed. This implies that the parameters $\mathbf{k}_{iR}$, $(\mathbf{r}_{,i})_R$ at such a point are equal to $\mathbf{k}^*_i$, $\mathbf{r}^*_{,i}$ and define the initial local shape of the surface—the same as $\mathbf{k}_i$, $\mathbf{r}_{,i}$. But the $\mathbf{k}_{iR}$, $(\mathbf{r}_{,i})_R$

must take into account the rotation of a locality of the surface (caused by deformation of its other parts). Thus the components of the $\mathbf{k}_{iR}$, $(\mathbf{r}_{,iR})$ in the rotated basis $\mathbf{n}^*$, $\mathbf{t}'_j$ are equal to the components of the initial form parameters $\mathbf{k}_i$, $\mathbf{r}_{,i}$ in the initial basis $\mathbf{n}$, $\mathbf{t}_j$:

$$\frac{\mathbf{k}_{iR}}{\alpha_i} = \frac{\mathbf{n}^* \times \mathbf{t}'_j}{R_{ij}} + \frac{\mathbf{n}^*}{R_{i3}}, \quad (\mathbf{r}_{,i})_R = \alpha_i \mathbf{t}'_i. \tag{7}$$

We consider the deformation of the surface at an arbitrary point M. Without loss of generality the analysis of strain may be simplified by taking the tangent plane at the point M as a reference for the displacement. This makes the rotated basis at the point M identical to the initial basis: $\mathbf{t}'_j(M) = \mathbf{t}_j(M)$, $\mathbf{n}^*(M) = \mathbf{n}(M)$. Hence the rotated local shape parameters, as defined in (7), remain at M equal to the initial surface shape parameters

$$\mathbf{k}_{iR}(M) = \mathbf{k}_i(M), \qquad (\mathbf{r}_{,i})_R = \mathbf{r}_{,i}(M). \tag{8}$$

Thus the application of the derivation formulas (3), (4) to the expressions (7) results for the chosen point M in

$$\mathbf{k}_{iR,j} = \mathbf{k}_{i,j} + \alpha_j \boldsymbol{\kappa}_j \times \mathbf{k}_i, \quad (\mathbf{r}_{,i})_{R,j} = \mathbf{r}_{,ij} + \alpha_j \boldsymbol{\kappa}_j \times \mathbf{r}_{,i}. \tag{9}$$

Now we introduce the expressions (6) into the eqn (5) and into $\mathbf{r}^*_{,12} = \mathbf{r}^*_{,21}$. Taking into account the derivation formulas (9), the relations (8) and, finally, using eqns (5) for the undeformed state and $\mathbf{r}_{,12} = \mathbf{r}_{,21}$ yields two vector compatibility equations

$$\begin{aligned} &(\alpha_1 \boldsymbol{\kappa}_1)_{,2} - (\alpha_2 \boldsymbol{\kappa}_2)_{,1} + \alpha_1 \alpha_2 \boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2 = 0, \\ &(\alpha_1 \boldsymbol{\epsilon}_1)_{,2} - (\alpha_2 \boldsymbol{\epsilon}_2)_{,1} - \alpha_1 t'_1 \times \alpha_2 \boldsymbol{\kappa}_2 + \alpha_2 t'_2 \times \alpha_1 \boldsymbol{\kappa}_1 = 0. \end{aligned} \tag{10}$$

With $\alpha_i \mathbf{t}'_i$ (not $\alpha_i \mathbf{t}_i$, as at the point M) the eqns (10) are written for any arbitrarily chosen reference system for displacements. Obviously, a change of the reference system for the displacement conforms to a rigid body displacement of the entire surface. This cannot influence in any way the relation between the strain parameters.

The vector parameters of extension-shear and of bending-twisting are most conveniently expressed in the rotated basis:

$$\boldsymbol{\epsilon}_i = \epsilon_{i1} \mathbf{t}'_1 + \epsilon_{i2} \mathbf{t}'_2, \qquad \boldsymbol{\kappa}_i = \mathbf{n}^* \times \mathbf{t}'_j \kappa_{ij} + \mathbf{n}^* \kappa_{i3}. \tag{11}$$

The geometric meaning of the scalar parameters introduced in (11) follows from the relations (6).

The parameters of curvature and twist of the deformed shell, appearing in the scalar compatibility equations in the components of the derivatives, are determined by the expressions (1), (6), (11) as

$$\frac{1}{R'_{in}} = \frac{1}{R_{in}} + \kappa_{in} \quad (i = 1, 2;\ n = 1, 2, 3). \tag{12}$$

The actual curvatures $1/R^*_{in}$ of the deformed surface are somewhat different, but can also be determined with the relations presented.

4. CONCLUDING REMARKS

Naturally the above analysis renders also the metric and the second fundamental tensor components a_{ij}, b_{ij}. The definitions $a^*_{ij} = \mathbf{r}^*_{,i} \cdot \mathbf{r}^*_{,j}$, $b^*_{ij} = \mathbf{n}^* \cdot \mathbf{r}^*_{,ij}$ together with the formulas (6), (7), (11) and $\mathbf{n}^* \cdot \mathbf{t}'_j = 0$ yield

$$a^*_{ij} = \alpha_i \alpha_j (\mathbf{t}'_i + \boldsymbol{\epsilon}_i) \cdot (\mathbf{t}'_j + \boldsymbol{\epsilon}_j), \qquad b^*_{ij} = \alpha_i \mathbf{n}^* \cdot [\mathbf{k}^*_j \times (\mathbf{t}'_i + \boldsymbol{\epsilon}_i)]. \tag{13}$$

The components a_{ij}, b_{ij} for the undeformed surface are determined by formulas (13) with

$$\mathbf{t}'_i + \boldsymbol{\epsilon}_i = \mathbf{t}_i,\ \mathbf{n}^* = \mathbf{n},\ \mathbf{k}^*_j = \mathbf{k}_j.$$

The geometric meaning of the parameters $\boldsymbol{\epsilon}_i$, R_{ij} is illustrated by the expressions (13) written for the simplest (and most important) case of small strain ($\mathbf{t}'_i + \boldsymbol{\epsilon}_i \approx \mathbf{t}'_i$) and for orthogonal coordinates $\xi_i(\chi = \pi/2)$:

$$\frac{a^*_{ij} - a_{ij}}{2\alpha_i\alpha_j} = \frac{1}{2}\epsilon_{ij} + \frac{1}{2}\epsilon_{ji}, \qquad \frac{b^*_{ij}}{\alpha_i\alpha_j} = -\frac{1+\epsilon_{ii}}{R'_{ji}} = -(1+\epsilon_{ii})\frac{1+\epsilon_{ii}}{R^*_{ji}}. \tag{14}$$

The six scalar compatibility equations resulting from the equations (10) coincide with the nonlinear equations of E. Reissner's work[1] if the decompositions (11) are supplemented with transverse shear terms $\epsilon_{i3}\mathbf{n}^*$ and the vectors $\mathbf{t}'_j$ are left in arbitrary position to $\mathbf{t}^*$.

The eqns (5) extend the vector Gauss–Codazzi equations[5] to oblique coordinates ξ_i and to arbitrarily deformed surfaces.

The vector compatibility eqns (10) are reduced to the known linear ones[1–3], when the nonlinear term $\alpha_i\alpha_2\boldsymbol{\kappa}_1 \times \boldsymbol{\kappa}_2$ is dropped, the shape of the surface is assumed to be identical with the initial shape ($\mathbf{n}^* = \mathbf{n}, \mathbf{t}'_j = \mathbf{t}_j$) and the coordinates ξ_j are orthogonal.

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